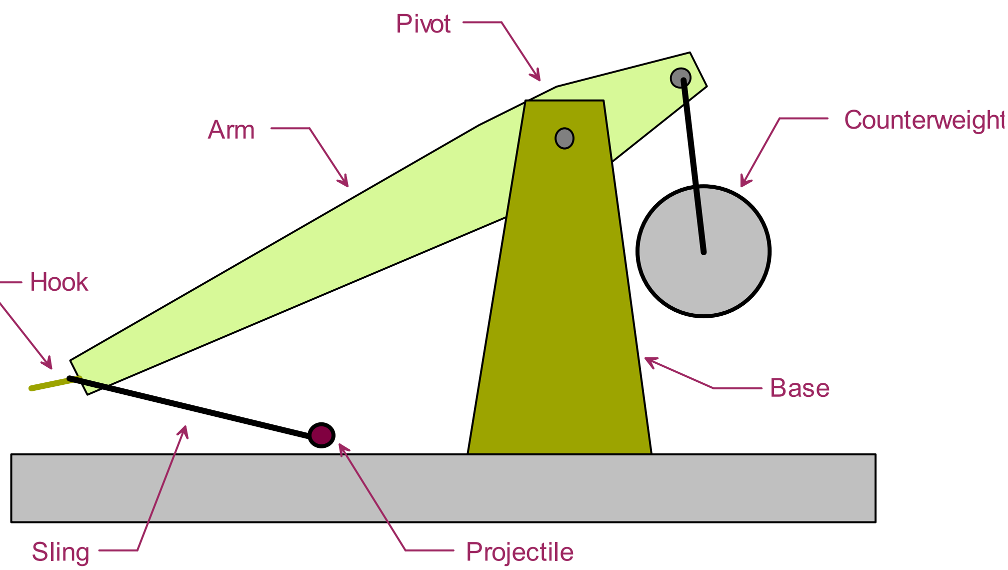
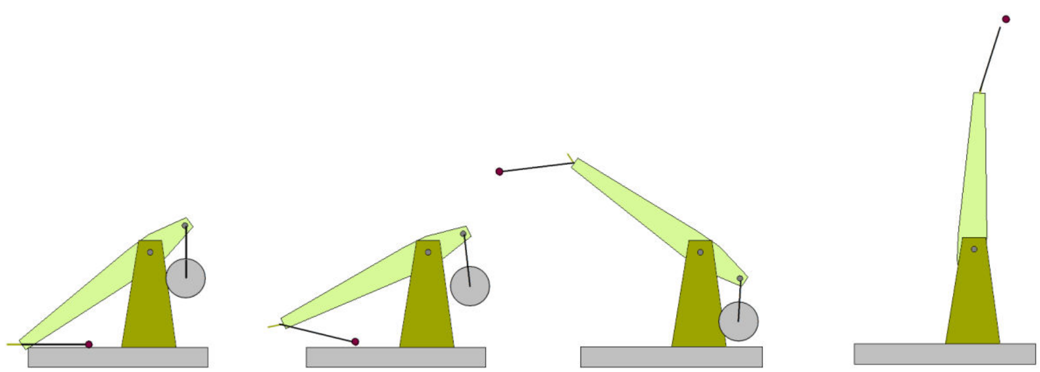


How to Simulate a Trebuchet Part 1: Lagrange's Equations

The trebuchet has quickly become a favorite project for physics and engineering teachers seeking to provide students with a simple – but spectacular – hands-on design project. The trebuchet itself is a simple device, easy (and forgiving) to build, and entirely powered by gravity. The major parts of the traditional trebuchet are shown schematically in the figure above. The *projectile* is held in a *sling* suspended from the *hook* on one side of the trebuchet *arm*. The *counterweight* is hung on the other (usually shorter) side of the arm. The arm rotates on a *pivot* which is rigidly mounted onto the *base*.



To launch the projectile, the counterweight is raised until the hook rests on the ground and then released. As the counterweight falls, the arm rotates (clockwise, in the diagram above) and the sling whips around until the projectile is released. The angle of the hook is critical here – the projectile is released when the sling and hook are aligned. The series of "cartoons" above illustrates this principle.

The purpose of this paper is to develop a dynamic model of the trebuchet that is easy to implement in software, whether it be in Matlab, C++, or (as with the website) Javascript. We will use the Lagrangian approach to dynamic systems, which involves developing expressions for the kinetic and potential energy of the system at a given configuration. The website contains models of several different trebuchets, from the very simple 1DOF model to the fully-developed floating-arm trebuchet. In this paper we will trace the development of the 3DOF model (the traditional trebuchet) in great detail. The important results for the other trebuchets are given at the end of the paper, and the interested reader can consult the appendices for details of their derivation.

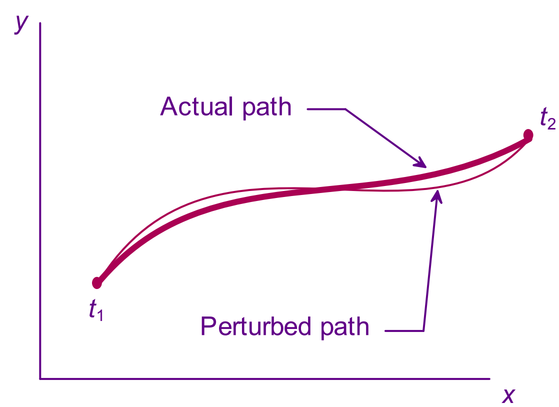
The Lagrangian approach to dynamics has the advantage of being relatively simple to implement, even if the derivation of Lagrange's equation is nontrivial (and the subject of most first-year graduate dynamics courses). The interested reader can refer to Ginsberg [1] for a clear development of the method or to Lanczos [2] for much of the philosophy behind the approach. The method for handling constraints used here is presented by Haug [3] and is the same as that used in many commercial codes. In my opinion, the presentation of Lanczos is the clearest of the lot, and is an excellent starting point for the study of analytical dynamics.

This paper is not the place for a full development of the Lagrange method, but it is worth noting that most explanations of the method seem to err on the side of being either too general and philosophical for immediate application, or too cumbersome for intelligibility. Here I have tried to present the method as it is applied to a specific problem, the trebuchet, in the hopes that the reader might find it easier to move from the specific to the general, rather than vice versa. People seem to learn best through induction, and that is the approach I have chosen here.

Lagrange's Equation – The Short Version

I present here a very short derivation of Lagrange's equation so as to familiarize the reader with the notation and the method. The development starts by stating Hamilton's principle, which is derived from the principle of virtual work. We define the "action integral", which computes the total difference between kinetic and potential energy in a system over the time interval t_1 to t_2.

$$A = \int_{t_1}^{t_2} (T - V)\, dt \tag{1}$$



Consider the path of a single particle as it moves between times t_1 and t_2 as shown in the bold curve in the figure above. Now imagine perturbing that path slightly, so that it lies along the fainter line. Note that we only perturb the path *between* the two endpoints; the positions at the endpoints must be the actual positions. What is the effect of this perturbation on the action integral for the particle?

$$\delta A = \delta \int_{t_1}^{t_2} (T - V)\, dt \tag{2}$$

Here we introduce the δ-operator, which defines the *variation* of a quantity. In this case, we are interested in the variation in the action integral produced by small perturbations of the paths of the particles. The δ operator is very similar to the d operator in differential calculus, with the exception that time is held constant with the δ operator. In differential calculus, the minimum (or maximum) value of a function is found when its derivative is zero. In similar fashion the action integral takes on a stationary (usually minimum) value when evaluated for the actual path, so that the variation of the action integral is zero for the actual path. Thus, Hamiltion's principle can be written

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \tag{3}$$

The commutative property of the variation process can be used to write

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0 \tag{4}$$

The potential energy of the system, V, is a function of the positions of the bodies in the system:

$$V = V(q_1, q_2, \dots, q_n) \tag{5}$$

where we use q to denote the *generalized coordinates* of the system. Each q might correspond to the angular position of one part of the trebuchet, or the height of another part. In contrast, the kinetic energy is a function of both the positions and velocities of the system

$$T = T(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n) \tag{6}$$

In general, the potential and kinetic energies might also be explicit functions of time, but this is not the case for the trebuchet. Using the chain rule (which also applies to the δ operator), the variations of the kinetic and potential energies can be written

$$\delta V = \frac{\partial V}{\partial q_1}\delta q_1 + \frac{\partial V}{\partial q_2}\delta q_2 + \cdots + \frac{\partial V}{\partial q_n}\delta q_n \tag{7}$$

$$\delta T = \frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \cdots + \frac{\partial T}{\partial q_n}\delta q_n + \frac{\partial T}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\delta \dot{q}_2 + \cdots + \frac{\partial T}{\partial \dot{q}_n}\delta \dot{q}_n \tag{8}$$

Combining these two terms and using summation notation gives the integral

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial V}{\partial q_i} \delta q_i \right) dt = 0 \tag{9}$$

The second term in the integral may be simplified slightly by using integration by parts, which will eliminate the $\delta \dot{q}_i$ term from the integral:

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i dt = \frac{\partial T}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i dt \tag{10}$$

Recall that the perturbations in the coordinates are zero at times t_1 and t_2, so that we have:

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i dt \tag{11}$$

and the complete integral becomes

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial T}{\partial q_i} \delta q_i - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i - \frac{\partial V}{\partial q_i} \delta q_i \right) dt = 0 \tag{12}$$

The integration by parts has enabled us to collect similar terms of δq_i such that

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial V}{\partial q_i} \right] \delta q_i dt = 0 \tag{13}$$

It is convenient at this point to multiply the equation above by -1, to put it into the more standard, well-known form:

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} \right) \delta q_i dt = 0 \tag{14}$$

As noted previously, the variations in the paths of the q between times t_1 and t_2 are *arbitrary* – we apply fictitious perturbations to the paths of each body in order to assess the effect on the action integral of the system. The integral equation above must sum to zero, regardless of which functions we choose to perturb the q's. The only way to *ensure* that the integral sums to zero is to force each term inside the summation to be zero. Thus, we have a set of n equations, one for each q_i.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{15}$$

This is the famous Lagrange's equation, which applies to any dynamic system whose kinetic and potential energy functions (and constraint equations) are not explicit functions of time. In an unconstrained problem, we would be required to solve all n of these equations simultaneously. The solution is rather simple to implement in a "cookbook" fashion, as will be seen below.

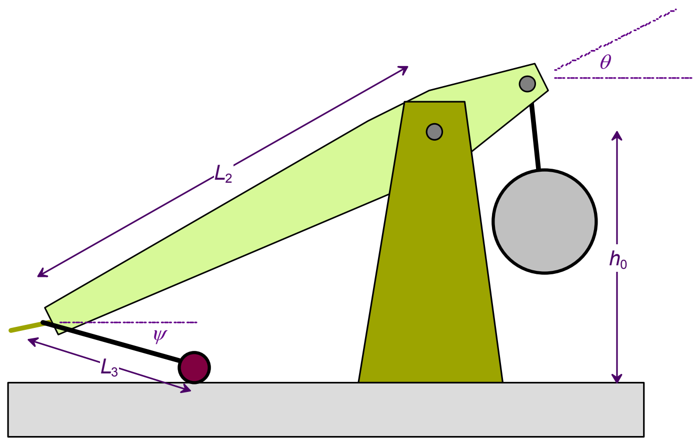


Figure 1: The projectile is in contact with the ground during the first few moments of launch.

How to Handle Constraints

The differential equations above are sufficient to model the motion of the unconstrained trebuchet. However, there are certain times during the motion of the trebuchet during which not all coordinates are independent. For example, the initial moments of the trebuchet launch has the projectile resting on the ground, as shown in Figure 1 above. In order that the projectile not pass through the ground, the angle of the sling, ψ, must be dependent upon the angle of the arm, θ, and vice versa. We can write a trigonometric relation between the two quantities as

$$-l_3 \sin\psi + l_2 \sin\theta = h_0 \tag{16}$$

where l_3 is the length of the sling, l_2 is the length of the arm (from pivot to hook) and h_0 is the height of the pivot. It would seem that we could use equation (16) to eliminate one of the variables, either ψ or θ, and reduce the system to n-1 degrees of freedom. In practice, however, this would result in a very complicated expression involving an inverse trigonometric function:

$$\psi = \sin^{-1}\left(\frac{l_2 \sin\theta - h_0}{l_3}\right) \tag{17}$$

Substitution of this into the equations of motion (and performing all the differentiations) would be cumbersome, to say the least. It is far more efficient to retain all of the generalized coordinates, keeping the problem at n degrees of freedom, and use another method to handle the constraints. Luckily, the Lagrange multiplier method is quite suited to the trebuchet problem.

The Method of Lagrange Multipliers

Let us first rewrite Equation 16 by moving all nonzero terms to the left-hand side

$$h_0 - l_2 \sin\theta + l_3 \sin\psi = 0 \tag{18}$$

For the trebuchet problem (and for many other problems in dynamics), it is always possible to write the constraint equations in this form:

$$f(q_1, q_2 \dots q_n) = 0 \tag{19}$$

In general (e.g. for the floating-arm trebuchet), we may have a set of constraint equations, instead of just one

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{Bmatrix} = \mathbf{0} \tag{20}$$

but for the present, let us assume that we have only one constraint equation. Generalization to *m* constraint equations is straightforward, once the method for handling constraints is known. If we perturb the generalized coordinates, then the variation of the constraint equation is found through the chain rule as

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f}{\partial q_n}\delta q_n = 0 \tag{21}$$

If we multiply both sides of this equation by an unknown multiplier, λ, we have

$$\lambda \cdot \delta f = \lambda\left(\frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 + \cdots + \frac{\partial f}{\partial q_n}\delta q_n\right) = 0 \tag{22}$$

where λ is known as a Lagrange multiplier. It may appear as though we have made our lives more complicated by introducing an unknown function, λ, but we will see that this method actually simplifies the constraint problem considerably. Collect all terms using summation notation.

$$\sum_{i=1}^{n} \lambda \frac{\partial f}{\partial q_i}\delta q_i = 0 \tag{23}$$

Since the above summation adds to zero, we may freely add it to the summation in Equation (14), since we are really just adding zero.

$$\int_{t_1}^{t_2} \left[\sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i - \frac{\partial T}{\partial q_i}\delta q_i + \frac{\partial V}{\partial q_i}\delta q_i\right) + \sum_{i=1}^{n}\lambda\frac{\partial f}{\partial q_i}\delta q_i\right] dt = 0 \tag{24}$$

but both summations are taken over the same indices, and so may be combined.

$$\int_{t_1}^{t_2} \sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i - \frac{\partial T}{\partial q_i}\delta q_i + \frac{\partial V}{\partial q_i}\delta q_i + \lambda\frac{\partial f}{\partial q_i}\delta q_i\right) dt = 0 \tag{25}$$

And finally, the common factor δq_i may be taken outside the parentheses.

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i} \right) \delta q_i \, dt = 0 \tag{26}$$

We apply the same argument as before: namely, that the only way to *guarantee* that the integral vanishes for arbitrary perturbations δq_i is to force each term of the summation to be zero.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i} = 0 \tag{27}$$

It probably seemed like a mysterious (and possibly useless) thing to do when we introduced the Lagrange multiplier in Equation (22). After all, what utility could adding zero to an equation possibly have? In the modified Lagrange's equation above, however, we see that the unknown multiplier has added an additional term to the left-hand side to account for the constraint equation.

The units on each of the terms in Equation (27) are those of force (or torque). But the derivative of the constraint equation is dimensionless, since we are differentiating a function of generalized coordinates (length) with respect to a generalized coordinate (length). Thus, the units of λ are also those of force (or torque). Although we will not prove it here, the Lagrange multipliers are proportional to the forces necessary to maintain the constraints. It is easy to generalize Equation (27) to account for multiple constraint equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \lambda_1 \frac{\partial f_1}{\partial q_i} + \lambda_2 \frac{\partial f_2}{\partial q_i} + \cdots + \lambda_m \frac{\partial f_m}{\partial q_i} = 0 \tag{28}$$

As before, we define $\mathbf{f}$ to be the vector of constraint equations, and $\boldsymbol{\lambda}$ the vector of Lagrange multipliers. Then the equation of motion for generalized coordinate *i* is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial \mathbf{f}}{\partial q_i}^T \boldsymbol{\lambda} = 0 \tag{29}$$

Matrix Form of the Equations of Motion

Let us examine more closely each of the terms in Equation 29. Expanding the time derivative in the first term gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_1 + \cdots \frac{\partial}{\partial q_n}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_n + \frac{\partial}{\partial \dot{q}_1}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\ddot{q}_1 + \cdots \frac{\partial}{\partial \dot{q}_n}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\ddot{q}_n \tag{30}$$

The first terms contain generalized coordinates and velocities, while the second term is a linear combination of the accelerations. Let us define the following quantities

$$\mathbf{m}_i = \left\{ \frac{\partial}{\partial \dot{q}_1}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \quad \frac{\partial}{\partial \dot{q}_2}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \quad \cdots \quad \frac{\partial}{\partial \dot{q}_n}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \right\} \tag{31}$$

$$g_i = -\frac{\partial}{\partial q_1}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_1 - \frac{\partial}{\partial q_2}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_2 - \cdots - \frac{\partial}{\partial q_n}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_n \tag{32}$$

Then the first term in Equation (29) can be written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \mathbf{m}_i\ddot{\mathbf{q}} - g_i \tag{33}$$

In similar fashion we define

$$h_i = \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} \tag{34}$$

Substituting these into Equation (29) gives the equation of motion for generalized coordinate *i.*

$$\mathbf{m}_i\ddot{\mathbf{q}} + \frac{\partial \mathbf{f}}{\partial q_i}^T \boldsymbol{\lambda} = g_i + h_i \tag{35}$$

There will be *n* of these equations, each corresponding to a single generalized coordinate $\mathbf{q}_i$. To solve the *n* equations simultaneously, we combine them into a single, matrix equation

$$\mathbf{M}\ddot{\mathbf{q}} + \frac{\partial \mathbf{f}}{\partial \mathbf{q}}^T \boldsymbol{\lambda} = \mathbf{g} + \mathbf{h} \tag{36}$$

where $\mathbf{M}$ is the inertia matrix of the system, $\partial\mathbf{f}/\partial\mathbf{q}$ is the Jacobian matrix of the constraint functions, and $\mathbf{g}$ and $\mathbf{h}$ are vectors of the remaining velocity and position terms after the differentiations in Equations (32) and (34) have been performed.

It appears that we have *n* equations of motion, but *n* + *m* unknowns (the set of $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$). We thus require *m* additional equations to solve the problem. Conveniently, the constraint equations (Eq. 20) provide just such a set. As discussed earlier, the nonlinearity of the constraint equations makes them difficult to use in eliminating variables. However, there is a trick we can use to get around this difficulty. By differentiating the constraint equations twice with respect to time, we can arrive at a set of constraint equations in terms of *accelerations* ($\ddot{\mathbf{q}}$) that are easy to solve. Since Equation (36) is also in terms of accelerations, we can combine the two sets for a single matrix equation.

Recall that the set of constraint equations is written

$$\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{Bmatrix} = \mathbf{0} \tag{37}$$

where each constraint equation is in terms of the generalized coordinates, $\mathbf{q}$. Taking the derivative of this equation with respect to time gives

$$\frac{d\mathbf{f}}{dt} = \frac{\partial\mathbf{f}}{\partial\mathbf{q}}\dot{\mathbf{q}} + \frac{\partial\mathbf{f}}{\partial t} = 0 \tag{38}$$

The second term goes to zero because (we assume) the contraints are not explicit functions of time. Thus

$$\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\dot{\mathbf{q}} = 0 \tag{39}$$

Differentiating once again gives the acceleration terms

$$\frac{d^2\mathbf{f}}{dt^2} = \frac{\partial}{\partial\mathbf{q}}\left(\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} + \frac{\partial\mathbf{f}}{\partial\mathbf{q}}\ddot{\mathbf{q}} = 0 \tag{40}$$

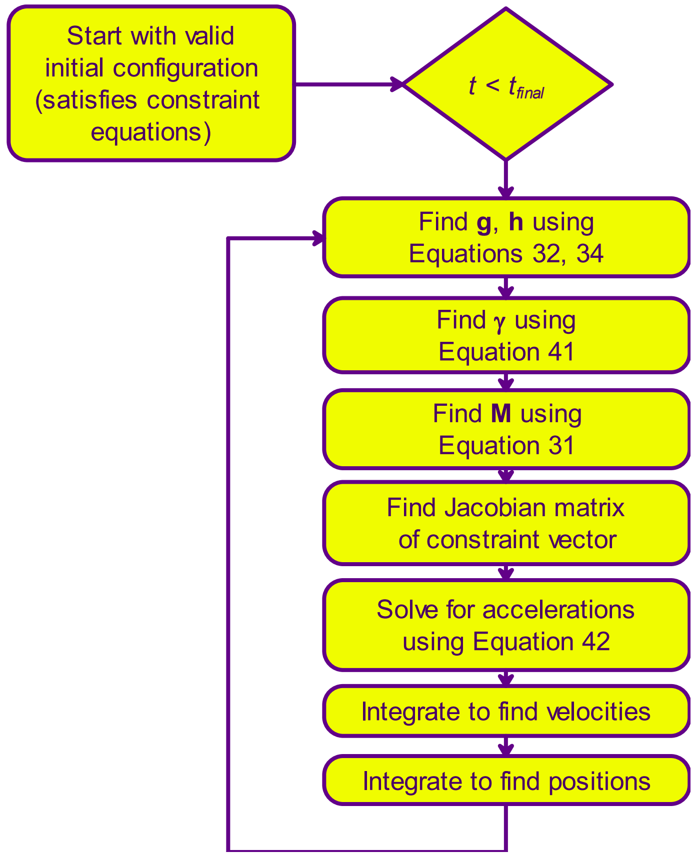
where again the partial derivatives with respect to time have been eliminated. We can solve for the accelerations

$$\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\ddot{\mathbf{q}} = -\frac{\partial}{\partial\mathbf{q}}\left(\frac{\partial\mathbf{f}}{\partial\mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \equiv \boldsymbol{\gamma} \tag{41}$$

and we combine Equations (36) and (41) to give

$$\begin{bmatrix} \mathbf{M} & \dfrac{\partial\mathbf{f}}{\partial\mathbf{q}}^T \\ \dfrac{\partial\mathbf{f}}{\partial\mathbf{q}} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} + \mathbf{h} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{42}$$

Since there are m constraints, we now have the $m + n$ equations that we require. These are the equations of motion that we solve to obtain the accelerations $\ddot{\mathbf{q}}$ and Lagrange multipliers $\boldsymbol{\lambda}$. After solving for the accelerations, we use numerical integration to find the velocities and positions of the bodies. The Lagrange multipliers are used to assess the current situation regarding the constraints. The flowchart below gives a procedure for implementing the solution on a computer.


Start with valid initial configuration (satisfies constraint equations)
t < t_final
Find g, h using Equations 32, 34
Find γ using Equation 41
Find M using Equation 31
Find Jacobian matrix of constraint vector
Solve for accelerations using Equation 42
Integrate to find velocities
Integrate to find positions